AN INVERSE PROBLEM FOR THE WAVE EQUATION WITH SOURCE AND RECEIVER AT DISTINCT POINTS

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Abstract. We consider the inverse problem of determining the density coefficient appearing in the wave equation from *separated* point source and point receiver data. Under some assumptions on the coefficients, we prove uniqueness results.

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1. Introduction

We address the inverse problem of determining the density coefficient of a medium by probing it with an external point source and by measuring the responses at a single point for a certain period of time.

More precisely, consider the following initial value problem (IVP), where $\Box = \partial_t^2 - \Delta_x$ denotes the wave operator:

$$\begin{aligned} (\Box - q(x))u(x,t) &= \delta(x,t), \quad (x,t) \in \mathbb{R}^3 \times \mathbb{R} \\ u(x,t)|_{t<0} &= 0, \qquad\qquad x \in \mathbb{R}^3. \end{aligned} \tag{1}$$

In Equation (1), we assume that the coefficient q is real-valued and is a $C^3(\mathbb{R}^3)$ function. The inverse problem we address is the unique determination of the coefficient q from the knowledge of $u(e,t)$ where $e = (1,0,0)$ for $t \in [0,T]$ with $T > 1$. Motivation for studying such problems arises in geophysics see [25] and references therein. Geophysicist determine properties of the earth structure by sending waves from the surface of the earth and measuring the corresponding scattered responses. Note that in the problem we consider here, the point source is located at the origin, whereas the responses are measured at a different point. Since the given data depends on one variable whereas the coefficient to be determined depends on three variables, some additional restrictions on the coefficient q are required to make the inverse problems tractable.

There are several results related to point source inverse problems involving the wave equation. We briefly mention here the details of works which are closely related to the problem studied in this article. Romanov in [18] considered the problem of determining the damping and density coefficients which are constant outside a bounded, simply connected domain $D \subset \mathbb{R}^3$. By using the expression for fundamental solution, he reduced the problem to an integral geometry problem (whose solution was known by [6]), which gives the determination of these coefficients in D when source and receiver are moving in a plane (say M) chosen in such a way that (i) $M \cap \overline{D} = \emptyset$ and (ii) the line segment joined by source and receiver lies completely outside $\overline{D}$. Rakesh in [12] studied the problem of determining q from the knowledge of $u(0,t)$ for $t \in [0,T]$ and he proved the uniqueness for coefficients which are either comparable or radially symmetric with respect to a point different from the source location. The above mentioned works are related to point source hyperbolic inverse problems with under-determined data. We also mention some related works for the point source hyperbolic inverse problems with formally determined or with over determined data. In [10] Rakesh proved the unique determination of the radially symmetric coefficient $q(x)$ appearing in (1) when $u(a,t)$ is known for all $a \in \mathbb{S}^2$ and $t \in [0,T]$. Rakesh and Sacks in [15] established the uniqueness for angular controlled coefficient appearing in (1) from the knowledge of $u(a,t)$ and $u_r(a,t)$ for all $a \in \mathbb{S}^2$ and

$t \in [0,T]$ where u_r denotes the derivative with respect to $r = |x|$. The problem considered in [15] can be seen as an extension of the work [10] to a set of more general coefficients which is strictly bigger than the set of radial functions but [15] requires more information than [10]. In [17] the problem of determining the density coefficient q with angular controlled is studied. They proved the uniqueness of these coefficients from the knowledge of $u^a(a,t)$ for all $a \in \mathbb{S}^2$ and $t \in [0,T]$, where $u^a(x,t)$ denote the solution to (1) when source is located at $a \in \mathbb{S}^2$. For more works related to the problem studied in this article, we refer to [3, 21, 16, 24, 18, 9, 13, 5, 8, 19] and references therein. For the one dimensional inverse problems related to the problem studied here, we refer to [1, 2, 14, 11].

We now state the main results of this article.

Theorem 1.1. *Suppose $q_i \in C^3(\mathbb{R}^3)$, $i = 1,2$ with $q_1(x) \geq q_2(x)$ for all $x \in \mathbb{R}^3$. Let $u_i(x,t)$ be the solution to the IVP*

$$\begin{aligned}(\Box - q_i(x))u_i(x,t) &= \delta(x,t), \qquad (x,t) \in \mathbb{R}^3 \times \mathbb{R}\\ u_i(x,t)|_{t<0} &= 0, \qquad\qquad\quad x \in \mathbb{R}^3.\end{aligned}$$

If $u_1(e,t) = u_2(e,t)$, for all $t \in [0,T]$ where $T > 1$ and $e = (1,0,0)$, then $q_1(x) = q_2(x)$ for all x with $|e-x| + |x| \leq T$.

Theorem 1.2. *Suppose $q_i \in C^3(\mathbb{R}^3)$, $i = 1,2$ with $q_i(x) = a_i(|x| + |x-e|)$ with $e = (1,0,0)$, for some C^3 functions a_i on $(1-\epsilon, \infty)$ for some $0 < \epsilon < 1$. Let u_i be the solution to the IVP*

$$\begin{aligned}(\Box - q_i(x))u_i(x,t) &= \delta(x,t), \qquad (x,t) \in \mathbb{R}^3 \times \mathbb{R}\\ u_i(x,t)|_{t<0} &= 0, \qquad\qquad\quad x \in \mathbb{R}^3.\end{aligned}$$

If $u_1(e,t) = u_2(e,t)$, for all $t \in [0,T]$ where $T > 1$ and $e = (1,0,0)$, then $q_1(x) = q_2(x)$ for all x with $|e-x| + |x| \leq T$.

Remark 1.3. From Proposition 2.1, the solution $u(x,t)$ of (1.1) is supported in $t \geq |x|$ hence $u(e,t) = 0$ for $t < 1$. So $u(e,t)$ has no information about q if $t < 1$, hence we require $T > 1$ in Theorems 1.1 and 1.2. Further note that ellipsoids $|x| + |x-e| \leq t$ are empty if $t < 1$.

To the best of our knowledge, our results, Theorems 1.1 and 1.2, which treat *separated* source and receiver, have not been studied earlier. Our result generalize the work [12], who considered the aforementioned inverse problem but with *coincident* source and receiver; see also [24].

The proofs of the above theorems are based on an integral identity derived using the solution to an adjoint problem as used in [22] and [24]. Recently this idea was used in [17] as well.

The article is organized as follows. In Section 2, we state the existence and uniqueness results for the solution of Equation (1), the proof of which is given in [4, 7, 19]. Section 3 contains the proofs of Theorems 1.1 and 1.2.

2. Preliminaries

Proposition 2.1. [4, pp.139,140] *Suppose q is a C^3 function on $\mathbb{R}^3$ and $u(x,t)$ satisfies the following IVP*

$$\begin{aligned}(\Box - q(x))u(x,t) &= \delta(x,t), \;\; (x,t) \in \mathbb{R}^3 \times \mathbb{R}\\ u(x,t)|_{t<0} &= 0, \qquad\qquad x \in \mathbb{R}^3\end{aligned} \tag{2}$$

then $u(x,t)$ is given by

$$u(x,t) = \frac{\delta(t-|x|)}{4\pi|x|} + v(x,t) \tag{3}$$

where $v(x,t)=0$ for $t<|x|$ and in the region $t>|x|$, $v(x,t)$ is a C^2 solution of the characteristic boundary value problem (Goursat Problem)

$$\begin{aligned} (\Box - q(x))v(x,t) &= 0, \qquad t>|x| \\ v(x,|x|) &= \frac{1}{8\pi}\int_0^1 q(sx)ds. \end{aligned} \tag{4}$$

We will use the following version of this proposition. Consider the following IVP

$$\begin{aligned} (\Box - q(x))U(x,t) &= \delta(x-e,t),\ \ (x,t)\in\mathbb{R}^3\times\mathbb{R} \\ U(x,t)|_{t<0} &= 0, \qquad\qquad x\in\mathbb{R}^3. \end{aligned} \tag{5}$$

Now we have

$$U(x,t)=\frac{\delta(t-|x-e|)}{4\pi|x-e|}+V(x,t) \tag{6}$$

where $V(x,t)=0$ for $t<|x-e|$ and for $t>|x-e|$, $V(x,t)$ is a C^2 solution to the following Goursat Problem

$$\begin{aligned} (\Box - q(x))V(x,t) &= 0, \qquad t>|x-e| \\ V(x,|x-e|) &= \frac{1}{8\pi}\int_0^1 q(sx+(1-s)e)ds. \end{aligned} \tag{7}$$

We can see this by translating source by $-e$ in Equation (5) and using the above proposition.

3. Proof of Theorems 1.1 and 1.2

In this section, we prove Theorems 1.1 and 1.2. We will first show the following three lemmas which will be used in the proof of the main results.

Lemma 3.1. *Suppose $q_i's$ $i=1,2$ be C^3 real-valued functions on $\mathbb{R}^3$. Let u_i be the solution to Equation (1) with $q=q_i$ and denote $u(x,t):=u_1(x,t)-u_2(x,t)$ and $q(x):=q_1(x)-q_2(x)$. Then we have the following integral identity*

$$u(e,\tau)=\int_{\mathbb{R}}\int_{\mathbb{R}^3} q(x)u_2(x,t)U(x,\tau-t)dxdt,\ \text{for all } \tau\in\mathbb{R} \tag{8}$$

where $U(x,t)$ is the solution to the following IVP

$$\begin{aligned} (\Box - q_1(x))U(x,t) &= \delta(x-e,t), \quad (x,t)\in\mathbb{R}^3\times\mathbb{R} \\ U(x,t)|_{t<0} &= 0, \qquad\qquad x\in\mathbb{R}^3. \end{aligned} \tag{9}$$

Proof. Since each u_i for $i=1,2$ satisfies the following IVP,

$$\begin{aligned} (\Box - q_i(x))u_i(x,t) &= \delta(x,t),\ \ (x,t)\in\mathbb{R}^3\times\mathbb{R} \\ u_i(x,t)|_{t<0} &= 0, \qquad\qquad x\in\mathbb{R}^3, \end{aligned}$$

we have that u satisfies the following IVP

$$\begin{aligned} (\Box - q_1(x))u(x,t) &= q(x)u_2(x,t),\ \ (x,t)\in\mathbb{R}^3\times\mathbb{R} \\ u(x,t)|_{t<0} &= 0, \qquad x\in\mathbb{R}^3. \end{aligned} \tag{10}$$

Now since

$$u(e,\tau)=\int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{R}} u(x,t)\delta(x-e,\tau-t)dtdx,$$

using (9), we have

$$u(e,\tau)=\int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{R}} u(x,t)(\Box-q_1(x))U(x,\tau-t)dtdx.$$

Now by using integration by parts and Equations (9) and (10), also taking into account that $u(x,t)=0$ for $t<|x|$ and that $U(x,t)=0$ for $|x-e|>t$, we get

$$\begin{aligned}u(e,\tau)&=\int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{R}} U(x,\tau-t)(\Box-q_1(x))u(x,t)dtdx\\&=\int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{R}} q(x)u_2(x,t)U(x,\tau-t)dtdx.\end{aligned}$$

This completes the proof of the lemma. $\square$

Lemma 3.2. *Suppose $q_i's$ are as in Lemma 3.1 and u_i is the solution to Equation (1) with $q=q_i$ and if $u(e,t):=(u_1-u_2)(e,t)=0$ for all $t\in[0,T]$, then there exists a constant $K>0$ depending on the bounds on v_2, V and T such that the following inequality holds*

$$\left|\int\limits_{|x-e|+|x|=2\tau}\frac{q(x)}{|2\tau x-|x|e|}dS_x\right|\leq K\int\limits_{|x-e|+|x|\leq 2\tau}\frac{|q(x)|}{|x||x-e|}dx,\ \forall\tau\in(1/2,T/2]. \tag{11}$$

Here dS_x is the surface measure on the ellipsoid $|x-e|+|x|=2\tau$ and v_2, V are solutions to the Goursat problem (see Equations (4) and (7)) corresponding to $q=q_i$.

Proof. From Lemma 3.1, we have

$$u(e,2\tau)=\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}^3} q(x)u_2(x,t)U(x,2\tau-t)dxdt,\ \text{for all } \tau\in\mathbb{R}.$$

Now since $u(e,2\tau)=0$ for all $\tau\in[0,T/2]$, and using Equations (3) and (6), we get

$$\begin{aligned}0=&\int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{R}} q(x)\frac{\delta(t-|x|)\delta(2\tau-t-|x-e|)}{16\pi^2|x||x-e|}dtdx\\&+\int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{R}} q(x)\frac{\delta(t-|x|)V(x,2\tau-t)}{4\pi|x|}dtdx\\&+\int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{R}} q(x)\frac{\delta(2\tau-t-|x-e|)}{4\pi|x-e|}v_2(x,t)dtdx\\&+\int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{R}} q(x)V(x,2\tau-t)v_2(x,t)dtdx.\end{aligned}$$

Now using the fact that $v_2(x,t)=0$ for $t<|x|$, $V(x,t)=0$ for $t<|x-e|$ and

$$\int\limits_{\mathbb{R}^n} \phi(x)\delta(P)dx = \int\limits_{P(x)=0} \frac{\phi(x)}{|\nabla_x P(x)|} dS_x$$

where dS_x is the surface measure on the surface $P=0$, we have that

$$\begin{aligned} 0 =& \frac{1}{16\pi^2} \int\limits_{|x-e|+|x|=2\tau} \frac{q(x)}{|x||x-e||\nabla_x(2\tau-|x|-|x-e|)|} dS_x \\ &+ \frac{1}{4\pi} \int\limits_{|x|+|x-e|\le 2\tau} \frac{q(x)V(x,2\tau-|x|)}{|x|} dx \\ &+ \frac{1}{4\pi} \int\limits_{|x|+|x-e|\le 2\tau} \frac{q(x)v_2(x,2\tau-|x-e|)}{|x-e|} dx \\ &+ \int\limits_{|x|+|x-e|\le 2\tau} \int\limits_{|x|}^{2\tau-|x-e|} q(x)V(x,2\tau-t)v_2(x,t)dtdx. \end{aligned}$$

For simplicity, denote

$$\begin{aligned} F(\tau,x) := \frac{1}{4\pi}\Big(&|x-e|V(x,2\tau-|x|) + |x|v_2(x,2\tau-|x-e|) \\ &+ 4\pi|x||x-e| \int\limits_{|x|}^{2\tau-|x-e|} V(x,2\tau-t)v_2(x,t)dt\Big) \end{aligned}$$

and using

$$|\nabla_x(2\tau-|x|-|x-e|)| = \left|\frac{x}{|x|} + \frac{x-e}{|x-e|}\right| = \left|\frac{|x-e|x+(x-e)|x|}{|x||x-e|}\right|.$$

We have

$$\frac{1}{16\pi^2} \int\limits_{|x-e|+|x|=2\tau} \frac{q(x)}{|2\tau x-|x|e|} dS_x = -\int\limits_{|x|+|x-e|\le 2\tau} \frac{q(x)}{|x||x-e|} F(\tau,x)dx.$$

Note that $\tau\in[0,T/2]$ with $T<\infty$. Now using the boundedness of v_2 and V on compact subsets, we have $|F(\tau,x)|\le K$ on $|x|+|x-e|\le T$.

Therefore, finally we have

$$\left|\int\limits_{|x-e|+|x|=2\tau} \frac{q(x)}{|2\tau x-|x|e|} dS_x\right| \le K \int\limits_{|x-e|+|x|\le 2\tau} \frac{|q(x)|}{|x||x-e|} dx,\ \forall \tau\in(1/2,T/2].$$

The lemma is proved. □

Lemma 3.3. *Consider the solid ellipsoid $|e-x|+|x|\leq r$, where $e=(1,0,0)$ and $x=(x_1,x_2,x_3)$, then we have its parametrization in prolate-spheroidal co-ordinates (ρ,θ,ϕ) given by*

$$\begin{aligned} x_1 &= \frac{1}{2}+\frac{1}{2}\cosh\rho\cos\phi \\ x_2 &= \frac{1}{2}\sinh\rho\sin\theta\sin\phi \\ x_3 &= \frac{1}{2}\sinh\rho\cos\theta\sin\phi \end{aligned} \tag{12}$$

with $\cosh\rho\leq r$, $\theta\in(0,2\pi)$, $\phi\in(0,\pi)$ *and the surface measure* dS_x *on* $|e-x|+|x|=r$ *and volume element* dx *on* $|e-x|+|x|\leq r$, *are given by*

$$\begin{aligned} dS_x &= \frac{1}{4}\sinh\rho\sin\phi\sqrt{\cosh^2\rho-\cos^2\phi}\,d\theta d\phi, \\ &\quad \text{with } \cosh\rho=r,\ \theta\in[0,2\pi] \text{ and } \phi\in[0,\pi] \\ dx &= \frac{1}{8}\sinh\rho\sin\phi(\cosh^2\rho-\cos^2\phi)d\rho d\theta d\phi, \\ &\quad \text{with } \cosh\rho\leq r,\ \theta\in[0,2\pi] \text{ and } \phi\in[0,\pi]. \end{aligned} \tag{13}$$

Proof. The above result is well known, but for completeness, we will give the proof. The solid ellipsoid $|e-x|+|x|\leq r$ in explicit form can be written as

$$\frac{(x_1-1/2)^2}{r^2/4}+\frac{x_2^2}{(r^2-1)/4}+\frac{x_3^2}{(r^2-1)/4}\leq 1.$$

From this, we see that

$$\begin{aligned} x_1 &= \frac{1}{2}+\frac{1}{2}\cosh\rho\cos\phi \\ x_2 &= \frac{1}{2}\sinh\rho\sin\theta\sin\phi \\ x_3 &= \frac{1}{2}\sinh\rho\cos\theta\sin\phi \end{aligned}$$

with $\cosh\rho\leq r$, $\theta\in[0,2\pi]$ and $\phi\in[0,\pi]$. This proves the first part of the lemma.

Now the parametrization of ellipsiod $|e-x|+|x|=r$, is given by

$$F(\theta,\phi)=\left(\frac{1}{2}+\frac{1}{2}\cosh\rho\cos\phi,\frac{1}{2}\sinh\rho\sin\theta\sin\phi,\frac{1}{2}\sinh\rho\cos\theta\sin\phi\right)$$

with $\theta\in(0,2\pi)$, $\phi\in(0,\pi)$ and $\cosh\rho=r$.

Next, we have

$$\begin{aligned} \frac{\partial F}{\partial\theta} &= \left(0,\frac{1}{2}\sinh\rho\cos\theta\sin\phi,-\frac{1}{2}\sinh\rho\sin\theta\sin\phi\right) \\ \frac{\partial F}{\partial\phi} &= \left(-\frac{1}{2}\cosh\rho\sin\phi,\frac{1}{2}\sinh\rho\sin\theta\cos\phi,\frac{1}{2}\sinh\rho\cos\theta\cos\phi\right). \end{aligned}$$

We have $dS_x = \left|\frac{\partial F}{\partial\theta}\times\frac{\partial F}{\partial\phi}\right| d\theta d\phi$, simple computation will gives us

$$dS_x = \frac{1}{4}\sinh\rho\sin\phi\sqrt{\cosh^2\rho-\cos^2\phi}\,d\theta d\phi,$$
$$\text{with } \cosh\rho = r,\ \theta\in(0,2\pi)\ and\ \phi\in(0,\pi).$$

Last part of the lemma follows from change of variable formula, which is given by

$$dx = \left|\frac{\partial(x_1,x_2,x_3)}{\partial(\rho,\theta,\phi)}\right| d\theta d\phi d\rho;\ \text{with } \cosh\rho\le r,\ \theta\in[0,2\pi] \text{ and } \phi\in[0,\pi]$$

where $\frac{\partial(x_1,x_2,x_3)}{\partial(\rho,\theta,\phi)}$ is given by

$$\frac{\partial(x_1,x_2,x_3)}{\partial(\rho,\theta,\phi)} = \det\begin{vmatrix} \frac{\partial x_1}{\partial\rho} & \frac{\partial x_1}{\partial\theta} & \frac{\partial x_1}{\partial\phi}\\ \frac{\partial x_2}{\partial\rho} & \frac{\partial x_2}{\partial\theta} & \frac{\partial x_2}{\partial\phi}\\ \frac{\partial x_3}{\partial\rho} & \frac{\partial x_3}{\partial\theta} & \frac{\partial x_3}{\partial\phi}\end{vmatrix}.$$

This gives

$$dx = \frac{1}{8}\sinh\rho\sin\phi(\cosh^2\rho-\cos^2\phi)\,d\rho d\theta d\phi,$$
$$\text{with } \cosh\rho\le r,\ \theta\in[0,2\pi] \text{ and } \phi\in[0,\pi].$$

□

3.1. **Proof of Theorem 1.1.** We first consider the surface integral in Equation (11) and denote it $Q(2\tau)$:

$$Q(2\tau) := \int\limits_{|x-e|+|x|=2\tau}\frac{q(x)}{|2\tau x-|x|e|}dS_x. \tag{14}$$

We have

$$\begin{aligned}|2\tau x-|x|e| &= |(2\tau x_1-|x|, 2\tau x_2, 2\tau x_3)| = \sqrt{(2\tau x_1-|x|)^2+4\tau^2x_2^2+4\tau^2x_3^2}\\ &= \sqrt{4\tau^2|x|^2+|x|^2-4\tau x_1|x|}.\end{aligned}$$

From Equation (12) and using the fact that $\cosh\rho = 2\tau$, we have

$$\begin{aligned}|2\tau x-|x|e| &= \frac{1}{2}\sqrt{(2\tau+\cos\phi)\{(4\tau^2+1)(2\tau+\cos\phi)-4\tau(1+2\tau\cos\phi)\}}\\ &= \frac{1}{2}\sqrt{(2\tau+\cos\phi)(8\tau^3+4\tau^2\cos\phi+2\tau+\cos\phi-4\tau-8\tau^2\cos\phi)}\\ &= \frac{1}{2}\sqrt{(2\tau+\cos\phi)(8\tau^3-4\tau^2\cos\phi-2\tau+\cos\phi)}\\ &= \frac{1}{2}\sqrt{(4\tau^2-\cos^2\phi)(4\tau^2-1)}.\end{aligned}$$

Using the above expression for $|2\tau x-|x|e|$ and Equation (13), we get

$$Q(2\tau) = \frac{1}{8}\int\limits_0^{\pi}\int\limits_0^{2\pi}\frac{q(\rho,\theta,\phi)\sinh\rho\sin\phi\sqrt{\cosh^2\rho-\cos^2\phi}}{\sqrt{(4\tau^2-\cos^2\phi)(4\tau^2-1)}}d\theta d\phi,$$

where we have denoted

$$q(\rho,\theta,\phi)=q\left(\frac{1}{2}+\frac{1}{2}\cosh\rho\cos\phi,\frac{1}{2}\sinh\rho\sin\theta\sin\phi,\frac{1}{2}\sinh\rho\cos\theta\sin\phi\right).$$

After using $\cosh\rho=2\tau$, $\sinh\rho=\sqrt{4\tau^2-1}$ and $\rho=\ln(2\tau+\sqrt{4\tau^2-1})$, we get

$$Q(2\tau)=\int_0^{\pi}\int_0^{2\pi} q(\ln(2\tau+\sqrt{4\tau^2-1}),\theta,\phi)\sin\phi d\theta d\phi. \tag{15}$$

Now consider the integral

$$\int_{|x-e|+|x|\leq 2\tau}\frac{q(x)}{|x||x-e|}dx.$$

Again using Equations (12) and (13) in the above integral, we have

$$\int_{|x-e|+|x|\leq 2\tau}\frac{q(x)}{|x||x-e|}dx=\frac{1}{2}\int_{\cosh\rho\leq 2\tau}\int_0^{\pi}\int_0^{2\pi} q(\rho,\theta,\phi)\sinh\rho\sin\phi d\theta d\phi d\rho.$$

After substituting $\cosh\rho=r$ and $\rho=\ln(r+\sqrt{r^2-1})$, we get

$$\int_{|x-e|+|x|\leq 2\tau}\frac{q(x)}{|x||x-e|}dx=\frac{1}{2}\int_1^{2\tau}\int_0^{\pi}\int_0^{2\pi} q(\ln(r+\sqrt{r^2-1}),\theta,\phi)\sin\phi d\theta d\phi dr.$$

Now using (15), we get

$$\int_{|x-e|+|x|\leq 2\tau}\frac{q(x)}{|x||x-e|}dx\leq C\int_1^{2\tau} Q(r)dr. \tag{16}$$

Now applying this inequality in Equation (11) and noting that $q(x)=q_1(x)-q_2(x)\geq 0$, we have

$$Q(2\tau)\leq CK\int_1^{2\tau} Q(r)dr. \tag{17}$$

Now Equation (17) holds for all $\tau\in[1/2,T/2]$ and since $q(x)\geq 0$, for all $x\in\mathbb{R}^3$, therefore using the Gronwall's inequality, we get

$$Q(2\tau)=0,\quad \tau\in[1/2,T/2].$$

Now from Equation (14), again using $q(x)\geq 0$, we have $q(x)=0$, for all $x\in\mathbb{R}^3$ such that $|x|+|x-e|\leq T$. The proof is complete.

3.2. **Proof of Theorem 1.2.** Again first, we consider the surface integral in (11) and denote it by $Q(2\tau)$:

$$Q(2\tau):=\int_{|x|+|x-e|=2\tau}\frac{q(x)}{|2\tau x-|x|e|}dS_x.$$

and $q(x) := a(|x| + |x-e|)$. Now from Equations (12), (13) and (15) and hypothesis $q_i(x) = a_i(|x| + |x-e|)$ of the theorem, we get

$$Q(2\tau) = \frac{1}{8}\int_0^{\pi}\int_0^{2\pi} a(2\tau)\sin\phi \, d\theta d\phi = \frac{\pi}{2}a(2\tau). \tag{18}$$

Now consider the integral

$$\int_{|x|+|x-e|\leq 2\tau} \frac{q(x)}{|x||x-e|}dx.$$

Again using (12) and (13) in the above integral, we have

$$\int_{|x-e|+|x|\leq 2\tau} \frac{q(x)}{|x||x-e|}dx = \frac{1}{2}\int_{\cosh\rho\leq 2\tau}\int_0^{\pi}\int_0^{2\pi} q(\rho,\theta,\phi)\sinh\rho\sin\phi \, d\theta d\phi d\rho.$$

After substituting $\cosh\rho = r$ and $\rho = \ln(r+\sqrt{r^2-1})$, we get

$$\begin{aligned}\left|\int_{|x-e|+|x|\leq 2\tau} \frac{q(x)}{|x||x-e|}dx\right| &= \left|\frac{1}{2}\int_1^{2\tau}\int_0^{\pi}\int_0^{2\pi} a(r)\sin\phi \, d\theta d\phi dr\right| \\ &\leq C\int_1^{2\tau}|a(r)|dr.\end{aligned}$$

Now using this inequality and Equation (18) in (11), we see

$$|a(2\tau)| \leq C\int_1^{2\tau}|a(r)|dr. \tag{19}$$

Now Equation (19) holds for all $\tau \in [1/2, T/2]$, so using the Gronwall's inequality, we have

$$a(2\tau) = 0, \text{ for } \tau \in [1/2, T/2].$$

Thus, we have $q(x) = 0$, for all $x \in \mathbb{R}^3$ such that $|x| + |x-e| \leq T$. This conclude the proof of Theorem 1.2.

3. Conclusion

In this paper, we studied an inverse problem for the wave equation with *separated* point source and point receiver data. Our approach is based on construction of spherical wave solution using the solution to a Goursat problem, combined with the solution to an adjoint problem, we ended up with an integral identity. Then using the prolate-spheroidal co-ordinates and Grownwall's inequality, we completed the proof of the main results.

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References

[1] K. P. Bube and R. Burridge; The one-dimensional inverse problem of reflection seismology. SIAM Rev. 25 (1983), no. 4, 497–559.
[2] A. L. Bukhgeĭm and M.V. Klibanov; Uniqueness in the large of a class of multidimensional inverse problems. (Russian) Dokl. Akad. Nauk SSSR 260 (1981), no. 2, 269–272.
[3] R. Burridge; The Gel'fand-Levitan, the Marchenko, and the Gopinath-Sondhi integral equations of inverse scattering theory, regarded in the context of inverse impulse-response problems. Wave Motion 2 (1980), no. 4, 305–323.
[4] F.G. Friedlander; The wave equation on a curved space-time. Cambridge University Press, Cambridge-New York-Melbourne, 1975. Cambridge Monographs on Mathematical Physics, No. 2.
[5] M.V. Klibanov; Some inverse problems with a 'partial' point source. Inverse Problems 21 (2005), no. 4, 1379–1390.
[6] M.M. Lavrentév; E.Yu. Derevtsov and V.A. Sharafutdinov; "On the determination of an optical body, situated in a homogeneous medium, from its images," Dokl. Akad. Nauk SSSR, 260, No. 4, 799-803 (1981).
[7] M.M. Lavrent'ev; V.G. Romanov and S.P. Shishat·skiĭ; Ill-posed problems of mathematical physics and analysis. Translated from the Russian by J. R. Schulenberger.
[8] S. Li; Estimation of coefficients in a hyperbolic equation with impulsive inputs. J. Inverse Ill-Posed Probl. 14 (2006), no. 9, 891–904.
[9] Rakesh; An inverse impedance transmission problem for the wave equation. Comm. Partial Differential Equations 18 (1993), no. 3-4, 583–600.
[10] Rakesh; Inversion of spherically symmetric potentials from boundary data for the wave equation. Inverse Problems 14 (1998), no. 4, 999–1007.
[11] Rakesh; Characterization of transmission data for Webster's horn equation. Inverse Problems 16 (2000), no. 2, L9–L24.
[12] Rakesh; Inverse problems for the wave equation with a single coincident source-receiver pair. Inverse Problems 24 (2008), no. 1, 015012, 16 pp.
[13] Rakesh; An inverse problem for a layered medium with a point source. Problems 19 (2003), no. 3, 497–506.
[14] Rakesh and P. Sacks; Impedance inversion from transmission data for the wave equation. Wave Motion 24 (1996), no. 3, 263–274.
[15] Rakesh and P. Sacks; Uniqueness for a hyperbolic inverse problem with angular control on the coefficients. J. Inverse Ill-Posed Probl. 19 (2011), no. 1, 107–126.
[16] Rakesh; W.W. Symes; Uniqueness for an inverse problem for the wave equation. Comm. Partial Differential Equations 13 (1988), no. 1, 87–96.
[17] Rakesh and G. Uhlmann; The point source inverse back-scattering problem. Analysis, complex geometry, and mathematical physics: in honor of Duong H. Phong, 279–289.
[18] V.G. Romanov; On the problem of determining the coefficients in the lowest order terms of a hyperbolic equation. (Russian. Russian summary) Sibirsk. Mat. Zh. 33 (1992), no. 3, 156–160, 220; translation in Siberian Math. J. 33 (1992), no. 3, 497–500.
[19] V.G. Romanov; Integral geometry and inverse problems for hyperbolic equations, volume 26. Springer Science and Business Media, 2013.
[20] V.G. Romanov and D.I. Glushkova; The problem of determining two coefficients of a hyperbolic equation. (Russian) Dokl. Akad. Nauk 390 (2003), no. 4, 452–456.
[21] P. Sacks and W.W. Symes; Uniqueness and continuous dependence for a multidimensional hyperbolic inverse problem. Comm. Partial Differential Equations 10 (1985), no. 6, 635–676.
[22] F. Santosa and W.W. Symes; High-frequency perturbational analysis of the surface point-source response of a layered fluid. J. Comput. Phys. 74 (1988), no. 2, 318–381.
[23] M.M. Sondhi; A survey of the vocal tract inverse problem: theory, computations and experiments. Inverse problems of acoustic and elastic waves (Ithaca, N.Y., 1984), 1–19, SIAM, Philadelphia, PA, 1984.
[24] P.D. Stefanov; A uniqueness result for the inverse back-scattering problem. Inverse Problems 6 (1990), no. 6, 1055–1064.

[25] W.W. Symes; The seismic reflection inverse problem. Inverse Problems 25 (2009), no. 12, 123008, 39 pp.

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